Tutorial 9 : Selected problems of Assignment 8

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Q1) (HW 8, Q4) Fix $\alpha \in [0,1)$, $x_0 \in [0,1]$, consider the iteration

$x_n := \alpha x_{n-1}(1-x_{n-1})$, for any $n \in \mathbb{N}$. Show that 0 is the

unique fixed point (i.e. If (x_n) converges to x satisfying $x = \alpha x(1-x)$, then $x=0$)

Also, show that it is attractive: for any $x_0 \in [0,1]$, $\lim_{n\to\infty} x_n = 0$.

Sol) Define $T:[0,1] \to \mathbb{R}$ by $Tx := \alpha x(1-x)$. Showing $T([0,1]) \subseteq [0,1]$:

T is smooth with $T'(x) = \alpha(1-2x)$; $T''(x) = -2\alpha \leq 0$.

$\therefore T$ achieve maximum at $\frac{1}{2}$ with $T(\frac{1}{2}) = \frac{\alpha}{4} < 1$. $\therefore Tx < 1, \forall x \in [0,1]$

Also, $Tx \geq 0, \forall x \in [0,1]$. $\therefore T([0,1]) \subseteq [0,1]$

Showing $T:[0,1] \to [0,1]$ is a contraction: take $\gamma = \max_{x\in[0,1]} |T'(x)| = \alpha < 1$

then $\forall x, x' \in [0,1]$, by MVT, $|Tx - Tx'| = |T'(\xi)||x-x'| \leq \gamma \cdot |x-x'|$

$\therefore T$ is a contraction with $T(0)=0$.

Showing 0 is the unique fixed point: Suppose (x_n) converges to x. Then

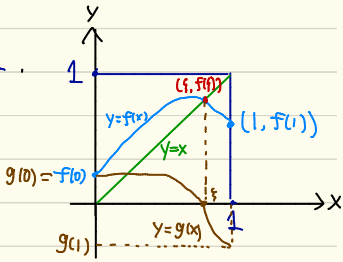
$Tx = x$, then by uniqueness part of Contraction Mapping Principle, $x=0$.

Also, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x_0 = 0$ by the existence part of Contraction Mapping Principle.

Q2) (HW8, Q5) Show that every continuous function $f:[0,1]\to[0,1]$ has a fixed point.

Sol) If $f(0)=0$ or $f(1)=1$, then f has a fixed point.

Otherwise, assume $f(0)>0$ and $f(1)<1$.



Define $g:[0,1]\to\mathbb{R}$ by $g(x)=f(x)-x$

then $g(0)>0$ and $g(1)<0$.

As g is continuous, by Intermediate Value Theorem,

there exists $\xi\in(0,1)$ s.t. $g(\xi)=0$, i.e. $f(\xi)=\xi$

Therefore, f has a fixed point.

Q3) (HW8, Q6) (Inverse Function Theorem for $\mathbb{R}$)

Let $f: [a,b] \to \mathbb{R}$ be a C^1 function. Show that

f admits a global differentiable inverse $g \iff \forall x \in (a,b),\ f'(x) \neq 0$

Sol) [$\Rightarrow$] Assuming such g exists, then $\forall x \in (a,b),\ g(f(x)) = x$;

differentiating both sides with respect to x : $g'(f(x)) \cdot f'(x) = 1$

$\therefore \forall x \in (a,b),\ f'(x) \neq 0$

[$\Leftarrow$] As $f': (a,b) \to \mathbb{R}$ is continuous, either $\forall x,\ f'(x) > 0$ or $\forall x,\ f'(x) < 0$

WLOG assume $\forall x,\ f'(x) > 0$; We first show that f is strictly increasing.

$\forall x, y \in [a,b],\ x < y$ by MVT, $\exists \xi \in (a,b)$ s.t. $f(y) - f(x) = f'(\xi)(y-x) > 0$

$\therefore f: [a,b] \to [f(a), f(b)]$ is strictly increasing continuous, in particular is bijective.

Then there exists global inverse $g: [f(a), f(b)] \to [a,b]$ which is continuous

and strictly increasing by the Continuous Inverse Theorem ([Bartle, Thm 5.6.5]).

Also, f is differentiable with $f'(x) \neq 0,\ \forall x \in (a,b)$. Therefore, by the

Differentiable Inverse Theorem ([Bartle: Thm 6.1.8]), g is also differentiable.